Step 3. $\zeta(1+it) \neq 0$.

Recall from Step 2 that for $c > 1$ (*strict* inequality), we have

$$\int_1^x \psi(t)\, dt = \frac{1}{2}x^2 - \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)\, x^{s+1} \frac{ds}{s\,(s+1)} + O(1)\,. \tag{25}$$

We hope to prove the Prime Number Theorem in the form $\psi(t) \sim t$. It can be shown that this implies $\int_1^x \psi(t)\, dt \sim x^2/2$. Thus we expect that the integral over the vertical line $c - i\infty$ to $c + i\infty$ in (25) to be an error term, i.e. smaller in magnitude than the main term. Yet the integrand contains a factor x^{s+1} which is of magnitude x^{c+1}. If $c > 1$ this term, x^{c+1}, is larger than the expected main term, $x^2/2$. So we can only have any hope of proving the Prime Number Theorem if we can choose $c = 1$. But, is F **well-defined** on the line $\mathrm{Re}\, s = 1$?

From its definition $F(s)$ has $\zeta(s)$ on the denominator which we know is non-zero for $\mathrm{Re}\, s > 1$, but is it non-zero **on** $\mathrm{Re}\, s = 1$?

Lemma 6.19 *For $\theta \in \mathbb{R}$, we have*

$$3 + 4\cos\theta + \cos 2\theta \geq 0.$$

Proof Start from the double angle formula

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1.$$

Then

$$\begin{aligned} 3 + 4\cos\theta + \cos 2\theta &= 2 + 4\cos\theta + 2\cos^2\theta \\ &= 2\,(1+\cos\theta)^2 \geq 0. \end{aligned}$$

■

Logarithms of complex numbers Recall that if $w \in \mathbb{C}$ then $w = re^{i\theta}$ for some $r \geq 0$ and θ. The logarithm of w is given by $\log w = \log r + i\theta$. But in fact $w = re^{i(\theta+2\pi k)}$ for *any* $k \in \mathbb{Z}$, in which case $\log w = \log r + i\,(\theta + 2\pi k)$. Thus the logarithm is **not** unique. Nonetheless, the logarithm of the modulus $|w|$ **is** unique and equals

$$\log|w| = \log r = \mathrm{Re}\,(\log r + i\,(\theta + 2\pi k)) = \mathrm{Re}\log w,$$

the real part of *any* logarithm of z. In particular, it can be shown from the Euler product for the Riemann zeta function that *a* logarithm of $\zeta(s)$ is given by

$$\log\zeta(s) = \sum_p -\log\left(1-\frac{1}{p^s}\right) = \sum_p \sum_{m=1}^{\infty} \frac{1}{mp^{ms}}.$$

Then using $\log|\zeta(s)| = \operatorname{Re}\log\zeta(s)$, we get

$$\log|\zeta(s)| = \operatorname{Re}\sum_p \sum_{m=1}^{\infty} \frac{1}{mp^{ms}} \tag{26}$$

for $\operatorname{Re} s > 1$. Yet

$$\operatorname{Re}\frac{1}{p^{ms}} = \operatorname{Re}\frac{e^{-itm\log p}}{p^{m\sigma}} = \frac{\cos(-mt\log p)}{p^{m\sigma}} = \frac{\cos(\theta_{m,t,p})}{p^{m\sigma}},$$

where $\theta_{m,t,p} = -mt\log p$. Hence

$$\log|\zeta(s)| = \sum_p \sum_{m=1}^{\infty} \frac{\cos(\theta_{m,t,p})}{mp^{m\sigma}}.$$

Lemma 6.20 *For $\sigma > 1$ we have*

$$|\zeta(\sigma)|^3\,|\zeta(\sigma+it)|^4\,|\zeta(\sigma+2it)| \geq 1.$$

Proof Consider

$$\begin{aligned}
&\log\left(|\zeta(\sigma)|^3\,|\zeta(\sigma+it)|^4\,|\zeta(\sigma+2it)|\right)\\
&\quad= 3\log|\zeta(\sigma)| + 4\log|\zeta(\sigma+it)| + \log|\zeta(\sigma+2it)|\\
&\quad= \sum_p \sum_{m=1}^{\infty}\left(\frac{3\cos(\theta_{m,0,p}) + 4\cos(\theta_{m,t,p}) + \cos(\theta_{m,2t,p})}{mp^{m\sigma}}\right).
\end{aligned}$$

Yet $\theta_{m,0,p} = 0$ and $\theta_{m,2t,p} = 2\theta_{m,t,p}$ and so this last expression equals

$$\sum_p \sum_{m=1}^{\infty}\left(\frac{3 + 4\cos(\theta_{m,t,p}) + \cos(2\theta_{m,t,p})}{mp^{m\sigma}}\right) \geq 0$$

by Lemma 6.19. Hence

$$|\zeta(\sigma)|^3\,|\zeta(\sigma+it)|^4\,|\zeta(\sigma+2it)| \geq e^0 = 1.$$

■

Theorem 6.21 *The Riemann zeta function has no zeros in the half-plane* $\mathrm{Re}\, s \geq 1$.

Proof We already know that $\zeta(s)$ is non-zero for $\mathrm{Re}\, s > 1$, so it remains only to prove there are no zeros on $\mathrm{Re}\, s = 1$.

Assume for contradiction that $\zeta(1 + it_0) = 0$ for some $t_0 \neq 0$ (recall that there is a pole at $s = 1$ and so is not zero there).

Recall from Theorem 6.12 that ζ is holomorphic at $1+it_0$ and thus its derivative exists there. By definition the derivative is a limit as $s \to 1+it_0$ along *any* path in $\mathbb{C}$. Choose the horizontal line to the right of $1 + it_0$ when $s = \sigma + it_0$ and $\sigma \to 1+$. Hence

$$\zeta'(1 + it_0) = \lim_{\sigma\to 1+} \frac{\zeta(\sigma+it_0) - \zeta(1 + it_0)}{\sigma-1} = \lim_{\sigma\to 1+} \frac{\zeta(\sigma+it_0)}{\sigma-1},$$

having used the assumption $\zeta(1 + it_0) = 0$.

From Theorem 6.12, we saw that $\zeta(s)$ has a simple pole at $s = 1$, residue 1, i.e.

$$\lim_{\sigma\to 1+} (\sigma-1)\, \zeta(\sigma) = 1.$$

Also, $\zeta(s)$ is holomorphic at $1 + 2it_0$, i.e. differentiable and thus continuous there, in which case

$$\lim_{\sigma\to 1+} \zeta(\sigma+2it_0) = \zeta(1 + 2it_0)\,.$$

We want to combine these three facts so consider, for $\sigma > 1$,

$$\begin{aligned} |(\sigma-1)\, \zeta(\sigma)|^3 \left|\frac{\zeta(\sigma+it_0)}{\sigma-1}\right|^4 |\zeta(\sigma+2it_0)| &= \frac{|\zeta(\sigma)|^3\, |\zeta(\sigma+it_0)|^4\, |\zeta(\sigma+2it_0)|}{\sigma - 1} \\ &\geq \frac{1}{\sigma-1}, \end{aligned} \tag{27}$$

by Lemma 6.20. Now let $\sigma \to 1+$, when the left had side of (27) tends to the finite limit

$$1^3\, |\zeta'(1 + it_0)|^4\, |\zeta(1 + 2it_0)|\,,$$

whilst the right hand side is unbounded.

This contradiction means that our original assumption was false, and thus $\zeta(s)$ has **no** zeros on $\mathrm{Re}\, s = 1$. ■

We can now see what was required in Lemma 6.19 for this proof to succeed. It was important that the coefficients in $3 + 4\cos\theta + \cos 2\theta$ were all positive integers and that the constant term 3, was less then at least one of the other coefficients, here 4.

In a later result we will show that $\zeta(s)$ in fact has no zeros slightly to the left of $\operatorname{Re} s = 1$.

Recall that zeros and poles of $\zeta(s)$ correspond with poles of the logarithmic derivative $\zeta'(s)/\zeta(s)$. Thus since $\zeta(s) \neq 0$ on $\operatorname{Re} s = 1$ we conclude that $\zeta'(s)/\zeta(s)$ has no poles on $\operatorname{Re} s = 1$, $s \neq 1$. We defined $F(s)$ as $\zeta'(s)/\zeta(s) + \zeta(s)$ because it has no pole at $s = 1$. Hence we conclude that $F(s)$ is well-defined on the closed half plane $\operatorname{Re} s \geq 1$.

Appendix for Step 3

This discussion comes from GJOJ pp 70-71

Recall that for $z \in \mathbb{C}$, **a** logarithm of z is **any** w for which $e^w = z$. So logarithms are not unique, they can differ by multiples of $2\pi i$. Assume that $z_j \in \mathbb{C}$ are given for $j \geq 1$, and logarithms w_j chosen for each z_j such that $\sum_j w_j$ converges to w. Then

$$\begin{aligned} e^w &= e^{\lim_{n\to\infty}\sum_{j=1}^n w_j} = \lim_{n\to\infty} e^{\sum_{j=1}^n w_j} \qquad \text{by the continuity of } e^z \text{ on } \mathbb{C}, \\ &= \lim_{n\to\infty}\prod_{j=1}^n e^{w_j} = \lim_{n\to\infty}\prod_{j=1}^n z_j \\ &= \prod_{j=1}^\infty z_j, \end{aligned}$$

by the definition of infinite products. Thus w is **a** logarithm of the product.

We can use this to find a logarithm of the ζ-function from

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1},$$

for $\mathrm{Re}\, s > 1$. the Euler product.

If we can find a logarithm for $(1 - 1/p^s)^{-1}$ for each prime p and show that the sum of the individual logarithms converges, **then** this lemma says that this sum of logarithms will be a logarithm of the Euler Product and thus of the Riemann zeta function.

Lemma 6.22

$$\sum_{m=1}^\infty \frac{z^m}{m}$$

*is **a** logarithm of* $1/(1-z)$ *for* $|z| < 1$.

Proof Let

$$h(z) = \sum_{m=1}^\infty \frac{z^m}{m}.$$

We need to show that

$$e^{h(z)} = \frac{1}{1-z}, \quad \text{i.e.} \quad (1-z)\, e^{h(z)} = 1.$$

Consider

$$\frac{d}{dz}(1-z)\,e^{h(z)} = -e^{h(z)} + (1-z)\,h'(z)\,e^{h(z)} = ((1-z)\,h'(z)-1)\,e^{h(z)}.$$

But

$$h'(z) = \sum_{m=1}^{\infty} m\frac{z^{m-1}}{m} = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z},$$

for $|z|<1$. Allowable since a power series can be differentiated term-by-term within its radius of convergence. Working back,

$$\frac{d}{dz}(1-z)\,e^{h(z)} = 0 \quad \text{i.e.} \quad (1-z)\,e^{h(z)} = c,$$

for some constant c. Put $z=0$ to see that $c=1$ as required. ■

Thus **a** logarithm of $(1-1/p^s)^{-1}$ is $\sum_{m=1}^{\infty} 1/mp^{ms}$. And the sum over p of these individual logarithms converges (absolutely) since

$$\sum_p \left|\sum_{m=1}^{\infty} \frac{1}{mp^{ms}}\right| \le \sum_p \sum_{m=1}^{\infty} \frac{1}{mp^{m\sigma}} \le \sum_p \sum_{m=1}^{\infty} \frac{1}{p^{m\sigma}} = \sum_p \frac{1}{p^{\sigma}-1},$$

having summed the inner geometric series, and this sum over primes converges for $\sigma>1$. Hence the Lemma tells us that

$$\sum_p \sum_{m=1}^{\infty} \frac{1}{mp^{ms}}$$

is **a** logarithm of $\zeta(s)$ for $\operatorname{Re} s > 1$.

Further, there is a general result

Theorem 6.23 *Assume that*

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |a_{ij}|$$

converges, then

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} \text{ and } \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}$$

both converge and to the same value S, say. Further, if $\sum_{n=1}^{\infty} c_n$ is any series obtained by rearranging the term a_{ij} as a single series, then it also converges to S.

In our case we rearrange the $a_{p,m} = 1/mp^{ms}$ in increasing order of p^m and deduce that the Dirichlet Series

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^s},$$

where $a(n) = 1/m$, if $n = p^m$ for some prime p, 0 otherwise, is convergent in $\operatorname{Re} s > 1$.